Fall 2019 Math 245 Final Exam

Please read the following directions:

Please write legibly, with plenty of white space. Please fill out the box above as legibly as possible. Please fit your answers in the designated areas. To get credit, you must also show adequate work to justify your answers. If unsure, show the work. All problems are worth 5-10 points. You may use a single page of notes, but no calculators or other aids. This exam will begin at 1:00 and will end at 3:00; pace yourself accordingly. Please remain quiet to ensure a good test environment for others. Good luck!

REMINDER: Use complete sentences.

Problem 1. Carefully define the following terms:

a. Double Negation semantic theorem

b. Modus Tollens semantic theorem

c. xor

Problem 2. Carefully define the following terms:

a. Vacuous Proof theorem

b. Proof by Contradiction theorem

c. big O

Problem 3. Let S, T be sets. Carefully state the converse of: If $S \subseteq T$, then $S \cup T = S$. Then, prove or disprove your statement.

REMINDER: Use complete sentences.

Problem 4. Carefully define the following terms:

a. Modular Division theorem

b. Chinese Remainder theorem

c. Order-Extension principle

Problem 5. Carefully define the following terms:

a. chain

b. range (or image)

c. identity function

Problem 6. Prove or disprove: $\forall n \in \mathbb{Z},\ \left\lfloor \frac{n}{2} \right\rfloor \geq \frac{n-1}{2}$.

Problem 7. Prove that the Fibonacci numbers satisfy: $\forall n \in \mathbb{N}_0,\ F_{n+2} = 1 + \sum_{i=0}^{n} F_i$.

Problem 8. Prove that there exists a unique set S, whose full relation R_{full} is irreflexive.
Note: you must prove both existence and uniqueness for S.

Problem 9. Define relation R on $\mathbb{Q}$ via $R = \{(a,b) : |a-b| \leq 3\}$. Prove that R is NOT an equivalence relation.

Problem 10. Prove or disprove: For all $x, y \in \mathbb{Z}$, if $x \equiv y \pmod{125}$, then $x \equiv y \pmod{25}$.

For problems 11,12: Define relation R on $\mathbb{R}$ via $R = \{(a,b) : \lfloor a \rfloor = \lfloor b \rfloor\}$.

Problem 11. With R as above, prove that R is an equivalence relation on $\mathbb{R}$.

Problem 12. With R as above, give, in set-builder notation, the equivalence class $\left[\frac{7}{3}\right]$.

Problem 13. Let R be a partial order on set S, $T \subseteq S$, and $a, a' \in T$. Suppose that a is a minimum for T, and a' is minimal in T. Prove that $a = a'$.

For problems 14,15: Consider the relation $|$ on $\mathbb{N}$, restricted to the interval poset $[3, 60]$.

Problem 14. Draw the Hasse diagram for the interval poset $[3, 60]$.

Problem 15. Find the width and height of the interval poset $[3, 60]$.

Problem 16. Define relation R on $\mathbb{N}$ via $R = \{(x, y) : y = 7 - x\}$. (i) Determine if R is left-total. (ii) Determine if R is right-definite. (iii) Determine if R is a function. Justify your answers.

Problem 17. Define $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\{(x, y) : xy = 1\}$. Prove that f is a bijection. NOTE: You need not prove that f is a function.

Problem 18. Let S_1, S_2, S_3 be sets, and $F_1 : S_1 \to S_2$ and $F_2 : S_2 \to S_3$ be functions. Suppose that F_1, F_2 are both surjective. Prove that $F_2 \circ F_1$ is surjective.
NOTE: Do not just cite a theorem, you are being asked to prove that theorem.

Problem 19. Consider the relation $|$ on $S = \{1, 2, 3, 4, 6, 8\}$. Find all linear extensions of $|$ on S.

Problem 20. Define $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Prove the following:

$$\forall \varepsilon \in \mathbb{R}^+ \ \exists \delta \in \mathbb{R}^+ \ \forall x \in \mathbb{R}, \ |x - 3| < \delta \to |5x - 15| < \epsilon.$$